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On The ES Spectrum and ES Energy of Graph

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Abstract

In 2024, Gutman *et al.* [19] introduced the molecular descriptor called the *Euler–Sombor* (ES) index to analyze the graphical structure of a molecule. We define the corresponding *Euler–Sombor matrix* M_{ES} for a graph Γ as follows: The entry at position $(i,j)^{th}$ is given by $\sqrt{{\rho_i}^2+{\rho_j}^2+\rho_i\rho_j}$ if the vertices μ_i and μ_j are adjacent, and 0 otherwise, where ρ_i denotes the degree of vertex μ_i. In this paper, we investigate the *Euler–Sombor* eigenvalues of some classes of graphs, and from these ES eigenvalues, we characterize the graphs. We establish bounds for the spectral radius of the graph and the Euler Sombor index of the graph and identify the graphs that attain these extremal values. We obtain the correlation coefficients of ES energy with some physicochemical properties of octane isomers.

Keywords: Euler-Sombor matrix, Euler-Sombor eigenvalue, second Zagreb index, Forgotten index, Euler-Sombor energy

1 Introduction

In this manuscript, every graph is assumed to be finite, simple and undirected. Let $\Gamma = (V, E)$ be a graph with vertex set $V(\Gamma) = \{\mu_1, \mu_2, \ldots, \mu_\gamma\}$ and

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edge set $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_m\}$. In the whole manuscript, if Γ consists of γ vertices, then we use K_γ, C_γ, P_γ and $K_{1,\gamma-1}$ to denote the complete graph, the cycle, the path, and the star graph on γ vertices, respectively. Furthermore, let $K_{\alpha,\beta}$ be the complete bipartite graph on the vertices α and β. Let Δ and δ be the maximum and minimum degrees of the graph, respectively. We write $\mu_i\mu_j \in E(\Gamma)$, if μ_i and μ_j are adjacent in Γ. We denote the degree of μ_i by ρ_i. For any concepts and notation-related results, the reader see Bapat [6] and Brouwer [9].

Topological indices are numerical descriptors that capture the structural characteristics of a molecule by interpreting it as a graph. These indices are useful in cheminformatics and computational chemistry because they describe how a molecule is connected, how it branches, and what its shape is like. They are used to predict a molecule's activity and properties, helping in drug and material development. In addition, they are used in chemical similarity searching, molecular design, and network analysis across fields like chemistry and biology. Topological indices remain unchanged under graph isomorphisms, implying that they are invariant under relabeling or specific transformations of the graph. Topological indices [17] provides a quantitative description of molecular structure, offering valuable insights into various chemical properties such as reactivity and stability. Prominent examples of such measures include the *Wiener index* [10], the *Zagreb indices* [20], and the *Randić index* [26]. Each of these indices captures distinct structural characteristics, making them powerful tools for investigating the physical and chemical behavior of molecules. Owing to their significance in both chemistry and mathematics, topological indices have been the focus of extensive research; see, for example, [2, 3, 4, 8, 11, 13, 23, 21, 32, 18, 27, 1]. The *second Zagreb index* is given by Gutman *et al.* [20] as,

$$SZ(\Gamma) = \sum_{\mu_i\mu_j \in E(\Gamma)} \rho_i\rho_j.$$

In 2015, for the graph Γ, the *Forgotten index* defined by Furtula *et al.* [14] as,

$$FI(\Gamma) = \sum_{\mu_i\mu_j \in E(\Gamma)} {\rho_i}^2 + {\rho_j}^2.$$

Gutman [17] introduced the vertex-based *Sombor index* as

$$SO(\Gamma) = \sum_{\mu_i\mu_j \in E(\Gamma)} \sqrt{{\rho_i}^2 + {\rho_j}^2}.$$

Recently, Gutman in 2024, proposed the *Euler-Sombor index* [19], having a close connection with the Sombor index, as

$$ES(\Gamma) = \sum_{\mu_i\mu_j \in E(\Gamma)} \sqrt{{\rho_i}^2 + {\rho_j}^2 + \rho_i\rho_j}.$$

The Euler–Sombor index has recently become an interesting topic for many researchers because it helps in understanding the structure of graphs and chemical compounds more effectively than several classical indices. Its applications in chemical graph theory, nanotechnology, and network analysis make it very useful. Researchers are also attracted to this index because it opens new directions for studying extremal graphs and other important mathematical properties.

Ren *et al.*, [29] studied the Euler–Sombor index of trees. The maximum Euler–Sombor index of unicyclic graphs with a given diameter was investigated in [30]. Raza *et al.*, [28] computed the Euler–Sombor index of maximal trees. Albalahi *et al.*, [1] optimized the Euler–Sombor index of molecular tricyclic graphs, while Gutman *et al.*, [24] introduced the Euler–Sombor index and its related variants.

Now using *Euler-Sombor index*, we define the Euler-Sombor matrix as follows:

Let Γ be a simple graph. The *Euler–Sombor matrix* $M_{ES} = [m_{ij}]_{\gamma\times\gamma}$ of Γ is the matrix whose $(i,j)^{th}$ entry defined as

$$m_{ij} = \begin{cases} \sqrt{{\rho_i}^2 + {\rho_j}^2 + \rho_i\rho_j}, & \text{if } \mu_i\mu_j \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases}$$

We denote the ν_1, ν_2, ..., ν_γ be the ES eigenvalues of M_{ES} matrix of Γ. Then Euler-Sombor energy $\mathcal{E}_{ES}$ is expressed as,

$$\mathcal{E}_{ES} = \sum_{i=1}^{\gamma} |\nu_i|.$$

For a graph Γ, we write $\Theta(\Gamma)$ for its *adjacency matrix*. This is an $\gamma \times \gamma$ matrix, where each entry is

$$\omega_{ij} = \begin{cases} 1, & \text{if } \mu_i\mu_j \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases}$$

If $\varphi_1, \varphi_2, \ldots, \varphi_\gamma$ are the eigenvalues of $\Theta(\Gamma)$, then the *energy* $\mathcal{E}_\Theta(\Gamma)$ of Γ is defined as $\mathcal{E}_\Theta(\Gamma) = \sum_{i=1}^{\gamma} |\varphi_i(\Gamma)|$.

In this paper, the second section outlines some basic preliminaries needed to study and discuss some properties of the ES spectrum of a graph. We obtain the ES eigenvalues of graphs such as regular graph, complete bipartite graph, complete split graph, and the ES characteristic polynomial of path graph and the characterization of graphs from their ES eigenvalues. Also, we establish the upper and lower bounds for the ES spectral radius of Γ. We obtain the bound on the ES index of Γ. We obtain the ES energy of some graphs in section 3. The last section contains the correlation coefficients of ES energy with physicochemical properties of octane isomers.

2 ES Spectrum of Some Graphs

Now, we begin by some well-known results from [6]. Lemmas 1, 2, 3 and 4 present the eigenvalues of the adjacency matrix $\Theta(\Gamma)$ for specific classes such as the complete graph K_γ, the complete bipartite graph $K_{\alpha,\beta}$, the cycle graph C_γ, and the star graph $K_{1,\gamma-1}$, respectively.

The Euler-Sombor matrix of a graph is an irreducible, symmetric, non-negative, real with all entries greater than or equal to zero, therefore its all eigenvalues are real. Also, we observe that the trace of $M_{ES}(\Gamma)$ is zero.

Lemma 1 ([6]). *For any integer $\gamma \geq 1$, the eigenvalues of K_γ are $\varphi_1 = \gamma - 1$, and $\varphi_2 = \cdots = \varphi_\gamma = -1$.*

Lemma 2 ([6]). *For any integer $\gamma \geq 2$, the eigenvalues of C_n are $\varphi_i = 2\cos(\frac{2\pi(i-1)}{\gamma})$, for $i = 1, 2, \ldots, \gamma$.*

Lemma 3 ([6]). *For any integer $\alpha, \beta \geq 1$, the eigenvalues of $K_{\alpha,\beta}$ are $\varphi_1 = \sqrt{\alpha\beta}$, $\varphi_2 = \cdots = \varphi_{\alpha+\beta-1} = 0$ and $\varphi_{\alpha+\beta} = -\sqrt{\alpha\beta}$.*

Lemma 4 ([9]). *For any integer $\gamma \geq 2$, the eigenvalues of $K_{1,\gamma-1}$ are $\varphi_1 = \sqrt{\gamma - 1}$, $\varphi_2 = \cdots = \varphi_{\gamma-1} = 0$ and $\varphi_\gamma = -\sqrt{\gamma - 1}$.*

The following results give the ES-eigenvalues of regular, complete bipartite graphs and the trace of $M_{ES}^2(\Gamma)$ in terms of the Forgotten and the second Zagreb indices of the graph.

Theorem 5. *Let Γ be a connected graph on $\gamma \geq 3$ vertices, and $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_\gamma$, and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_\gamma$ be the eigenvalues and ES eigenvalues of Γ.*

1. *If each vertex of* Γ *has degree* σ. *Then* $\nu_i = \sigma\sqrt{3}\varphi_i$ *for all* $1 \leq i \leq \gamma$. *In fact, if* $\Gamma = K_\gamma$. *Then* $\nu_1 = (\gamma-1)^2\sqrt{3}$ *and* $\nu_2 = \cdots = \nu_\gamma = -(\gamma-1)\sqrt{3}$. *Also, for* $\Gamma = C_\gamma$. *Then* $\nu_i = 2\sqrt{3}cos(\frac{2\pi(i-1)}{\gamma})$.

2. *If* $\Gamma = K_{\alpha,\beta}$, *with* $\gamma = \alpha + \beta$ *vertices. Then the ES eigenvalues of* $K_{\alpha,\beta}$ *are* $\pm\sqrt{\alpha\beta(\alpha^2+\beta^2+\alpha\beta)}$ *and* 0 *with multiplicity* $\alpha + \beta - 2$. *Also, if* $\Gamma = K_{1,\gamma-1}$. *Then the ES eigenvalues of* $K_{1,\gamma-1}$ *are* $\pm\sqrt{(\gamma-1)(\gamma^2-\gamma+1)}$ *and* 0 *with multiplicity* $\gamma - 2$.

3. $\sum_{i=1}^{\gamma}\nu_i^2(\Gamma) = 2(FI(\Gamma) + SZ(\Gamma))$ *and* $\sum_{1\leq i<j\leq\gamma}\nu_i\nu_j(\Gamma) = -(FI(\Gamma) + SZ(\Gamma))$.

Proof. 1. If Γ is an σ-regular graph of order γ. Then, it is easy to see that $M_{ES}(\Gamma) = \sigma\sqrt{3}\ \Theta(\Gamma)$. Thus $\nu_i = \sigma\sqrt{3}\ \varphi_i$ for $i = 1, 2, \ldots, \gamma$. Also, as K_γ and C_γ are $\gamma - 1$ and 2 regular graphs. Then from Lemmas 1 and 2, would yield the required result.

2. It is straightforward to observe that, $M_{ES}(K_{\alpha,\beta}) = (\sqrt{\alpha^2+\beta^2+\alpha\beta})$ $\Theta(K_{\alpha,\beta})$. From the above Lemma 3, the eigenvalues of $\Theta(K_{\alpha,\beta})$ are $\pm\sqrt{\alpha\beta}$ and 0 with multiplicity $\alpha+\beta-2$. Therefore, ES eigenvalues of $K_{\alpha,\beta}$ are $\pm\sqrt{\alpha\beta(\alpha^2+\beta^2+\alpha\beta)}$ and remaining are 0 with multiplicity $\alpha + \beta - 2$. Also, The star graph is the special case of $K_{\alpha,\beta}$. Also, $M_{ES}(K_{1,\gamma-1}) = (\sqrt{\gamma^2-\gamma+1})\Theta(K_{1,\gamma-1})$ and from Lemma 4, we have the ES eigenvalues of $K_{1,\gamma-1}$ are $\pm\sqrt{(\gamma-1)(\gamma^2-\gamma+1)}$ and 0 with multiplicity $\gamma - 2$.

3. We know that, $\sum_{i=1}^{\gamma}\nu_i = \text{trace}(M_{ES}(\Gamma)) = 0$.

$$\sum_{i=1}^{\gamma}\nu_i^2 = \text{trace}(M_{ES}^2(\Gamma)) = 2\sum_{\mu_i\mu_j\in E(\Gamma)}\left(\rho_i^2+\rho_j^2+\rho_i\rho_j\right)$$

$$= 2\left(\sum_{\mu_i\mu_j\in E(\Gamma)}(\rho_i^2+\rho_j^2) + \sum_{\mu_i\mu_j\in E(\Gamma)}\rho_i\rho_j\right) = 2(FI(\Gamma)+SZ(\Gamma)).$$

Moreover,

$$\sum_{1\leq i<j\leq\gamma}\nu_i\nu_j = \frac{1}{2}\left(\left(\sum_{i=1}^{\gamma}\nu_i\right)^2 - \sum_{i=1}^{\gamma}\nu_i^2\right) = -(FI(\Gamma)+SZ(\Gamma)).$$

□

In the next result, we obtain the ES characteristic polynomial of the path graph.

Theorem 6. *For $\gamma \geq 5$, let P_γ be a path graph on γ vertices. Then $\phi\big(P_\gamma : \nu\big) = \nu^2\Omega_{\gamma-2} - 14\nu\Omega_{\gamma-3} + 49\Omega_{\gamma-4}$, where $\Omega_1 = \nu$, $\Omega_2 = \nu^2 - 12$, and $\Omega_m = \nu\Omega_{m-1} - 12\Omega_{m-2}$, for $m \geq 3$.*

Proof. For $\gamma \geq 5$, let P_γ be a path graph on γ vertices. Then, the M_{ES} matrix of P_γ is

$$M_{ES}(P_\gamma) = \begin{pmatrix} 0 & \sqrt{7} & 0 & \dots & 0 \\ \sqrt{7} & 0 & \sqrt{12} & 0 & \vdots \\ 0 & \sqrt{12} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{7} \\ 0 & \dots & 0 & \sqrt{7} & 0 \end{pmatrix}_\gamma .$$

Therefore, the characteristic polynomial is

$$\phi\big(P_\gamma : \nu\big) = det(\mu I - M_{ES}(P_\gamma)) = \begin{vmatrix} \nu & -\sqrt{7} & 0 & \dots & 0 \\ -\sqrt{7} & \nu & -\sqrt{12} & 0 & \vdots \\ 0 & -\sqrt{12} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\sqrt{7} \\ 0 & \dots & 0 & -\sqrt{7} & \nu \end{vmatrix}_\gamma .$$

For any $m \geq 3$, consider

$$C_m = \begin{pmatrix} \nu & -\sqrt{12} & 0 & \dots & 0 \\ -\sqrt{12} & \nu & -\sqrt{12} & 0 & \vdots \\ 0 & -\sqrt{12} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\sqrt{12} \\ 0 & \dots & 0 & -\sqrt{12} & \nu \end{pmatrix}_m .$$

Let Ω_m be the $\det(C_m)$. Observe that, $\Omega_m = \det(C_m) = \nu\Omega_{m-1} - 12\Omega_{m-2}$. We easily prove by using the same technique which used in [7]. Therefore, $\phi\big(P_n(G) : \nu\big) = \nu^2\Omega_{\gamma-2} - 14\nu\Omega_{\gamma-3} + 49\Omega_{\gamma-4}$. □

If any two vertices are adjacent in Γ, then they form a clique, and otherwise it is an independent set in Γ. A graph is a complete split graph if

it can be partitioned into an independent set and a clique such that every vertex in the independent set is adjacent to every vertex in the clique. We denote the complete split graph by $CS_{\omega,\gamma-\omega}$, where ω and $\gamma-\omega$ are the number of vertices in cliques and independent set in Γ, respectively. The following result gives the ES-eigenvalues of the complete split graph.

Theorem 7. *The ES eigenvalues of complete split graph* $CS_{\omega,\gamma-\omega}$ *are* $\frac{(\omega-1)(\gamma-1)\sqrt{3} \pm \sqrt{3(\omega-1)^2(\gamma-1)^2 + 4\omega(\gamma-\omega)\left[(\gamma-1)^2+\omega^2+(\gamma-1)\omega\right]}}{2}$, $-(\gamma-1)\sqrt{3}$ *with multiplicity* $\omega-1$ *and* 0 *with multiplicity* $\gamma-\omega-1$.

Proof. Suppose $J=\{\mu_1,\mu_2,\ldots,\mu_\omega\}$ is a clique with ω vertices of $CS_{\omega,\gamma-\omega}$. As J is a clique then each vertex share the same neighbourhood, that is the degree of every vertices in J are equal, therefore we have $\rho_1=\rho_2=\cdots=\rho_\omega=(\gamma-1)\sqrt{3}$. We labeling first the vertices from J so that the ES matrix of $CS_{\omega,\gamma-\omega}$ can be written as

$$M_{ES}(CS_{\omega,\gamma-\omega}) = \left(\begin{array}{cccc|ccc} 0 & (\gamma-1)\sqrt{3} & \ldots & (\gamma-1)\sqrt{3} & \chi & \ldots & \chi \\ (\gamma-1)\sqrt{3} & 0 & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & (\gamma-1)\sqrt{3} & \vdots & \ddots & \vdots \\ (\gamma-1)\sqrt{3} & (\gamma-1)\sqrt{3} & \ldots & 0 & \chi & \ldots & \chi \\ \hline \chi & \ldots & \ldots & \chi & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi & \ldots & \ldots & \chi & 0 & \ldots & 0 \end{array}\right)$$

where $\chi=\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega}$.

$$M_{ES}(CS_{\omega,\gamma-\omega}) = \left(\begin{array}{cccc|c} 0 & (\gamma-1)\sqrt{3} & \ldots & (\gamma-1)\sqrt{3} & \\ (\gamma-1)\sqrt{3} & 0 & \ddots & \vdots & X_{\omega\times(\gamma-\omega)} \\ \vdots & \ddots & \ddots & (\gamma-1)\sqrt{3} & \\ (\gamma-1)\sqrt{3} & (\gamma-1)\sqrt{3} & \ldots & 0 & \\ \hline \multicolumn{4}{c|}{X_{(\gamma-\omega)\times\omega}} & O_{(\gamma-\omega)\times(\gamma-\omega)} \end{array}\right),$$

where O be the zero matrix of order $(\gamma-\omega)\times(\gamma-\omega)$. For $i=2,3,\ldots,\omega$, let $Y_{i-1}=(-1,y_{i2},y_{i3},\ldots,y_{i\omega}\underbrace{0,0,0,\ldots,0}_{\gamma-\omega})^T$ be the vector in $\mathbf{R}^\gamma$ such that

$y_{ij} = 1$ if $i = j$ and 0 otherwise. Suppose $Y_1, Y_2, \ldots, Y_{\omega-1}$ are linearly dependent vectors. Then there exist scalars $\beta_1, \beta_2, \ldots, \beta_{\omega-1}$ not all zero, such that

$$\beta_1 Y_1 + \beta_2 Y_2 + \cdots + \beta_{\omega-1} Y_{\omega-1} = \mathbf{0}.$$

This implies that

$$(-\sum_{i=1}^{\omega-1} \beta_i, \beta_1, \beta_2, \ldots, \beta_{\omega-1}, 0, 0, \ldots, 0) = \mathbf{0},$$

and it follows that $\beta_1 = \beta_2 = \cdots = \beta_{\omega-1} = 0$. Therefore, the vectors $Y_1, Y_2, \ldots, Y_{\omega-1}$ cannot be linearly dependent. Noting that rows of X are identical, we see that

$$M_{ES}(\Gamma)\, Y_1 = \left((\gamma-1)\sqrt{3}, -(\gamma-1)\sqrt{3}, 0, \ldots, 0, 0, \ldots, , 0 \right)^T = -(\gamma-1)\sqrt{3}\, Y_1.$$

Similarly, we see that $Y_2, Y_3, \ldots, Y_{\omega-1}$ are the eigenvectors of $M_{ES}(CS_{\omega,\gamma-\omega})$ corresponding to eigenvalue $-(\gamma-1)\sqrt{3}$ has multiplicity at least $\omega - 1$, since $-(\gamma-1)\sqrt{3}$ can repeated in the remaining eigenvalues of $M_{ES}(CS_{\omega,\gamma-\omega})$. Also, we observe that the $\gamma - \omega$ rows of $M_{ES}(CS_{\omega,\gamma-\omega})$ are identical. This implies that the rank of $M_{ES}(CS_{\omega,\gamma-\omega})$ is $\omega+1$. Therefore, 0 is the eigenvalue of $M_{ES}(CS_{\omega,\gamma-\omega})$ with multiplicity $\gamma - \omega - 1$.

Now, to obtain the remaining two eigenvalues of $M_{ES}(CS_{\omega,\gamma-\omega})$ we use the technique of an equitable matrix. We rewrite the $M_{ES}(CS_{\omega,\gamma-\omega})$ matrix in block matrix form as

$$M_{ES}(CS_{\omega,\gamma-\omega}) = \left(\begin{array}{c|c} A_{\omega\times\omega} & X_{\omega\times(\gamma-\omega)} \\ \hline X^T_{(\gamma-\omega)\times\omega} & O_{(\gamma-\omega)\times(\gamma-\omega)} \end{array} \right)_{\gamma\times\gamma.}$$

In the block matrix $A_{\omega\times\omega}$ all the diagonal entries are 0 and remaining all $\omega^2 - \omega$ non-diagonal entries are $(\gamma-1)\sqrt{3}$. Therefore, the block row sum is $(\omega^2-\omega)(\gamma-1)\sqrt{3}$. Then the entry in equitable quotient matrix is $\frac{(\omega^2-\omega)(\gamma-1)\sqrt{3}}{\omega}$. Also, same way for the matrix $X_{\omega\times(\gamma-\omega)}$ all the entries are $\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega}$. Therefore, the block row sum is $\omega(\gamma-\omega)\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega}$. Then the entry in the equitable quotient matrix is $\frac{\omega(\gamma-\omega)\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega}}{\omega}$. Similar manner we compute other entries in the equitable quotient matrix of $CS_{\omega,\gamma-\omega}$. We denote the equitable quotient matrix of $CS_{\omega,\gamma-\omega}$ by $Q(CS_{\omega,\gamma-\omega})$. The equitable quotient matrix of $CS_{\omega,\gamma-\omega}$ is written as follows.

$$Q(CS_{\omega,\gamma-\omega}) =$$
$$\begin{pmatrix} (\omega-1)(\gamma-1)\sqrt{3} & (\gamma-\omega)\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega} \\ \omega\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega} & 0 \end{pmatrix}.$$

Let μ_1 and μ_2 be two eigenvalues of $Q(CS_{\omega,\gamma-\omega})$. Now, the characteristic polynomial of $Q(CS_{\omega,\gamma-\omega})$ is

$$|Q(CS_{\omega,\gamma-\omega}) - \mu I| =$$
$$\begin{vmatrix} (\omega-1)(\gamma-1)\sqrt{3}-\mu & (\gamma-\omega)\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega} \\ \omega\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega} & -\mu \end{vmatrix} = 0.$$

After simplification we get,

$$\mu^2 - (\omega-1)(\gamma-1)\sqrt{3}\mu - (\gamma-\omega)\omega\sqrt{(\gamma-1)^2+\omega^2+(\gamma-1)\omega} = 0.$$

Therefore, the roots of polynomial are

$$\mu_1 = \frac{(\omega-1)(\gamma-1)\sqrt{3} + \sqrt{3(\omega-1)^2(\gamma-1)^2 + 4\omega(\gamma-\omega)\left[(\gamma-1)^2+\omega^2+(\gamma-1)\omega\right]}}{2} \text{ and}$$
$$\mu_2 = \frac{(\omega-1)(\gamma-1)\sqrt{3} - \sqrt{3(\omega-1)^2(\gamma-1)^2 + 4\omega(\gamma-\omega)\left[(\gamma-1)^2+\omega^2+(\gamma-1)\omega\right]}}{2}. \qquad \square$$

The next result, which is one of the special case of the above Theorem, it gives the ES eigenvalues of $K_\gamma - e$.

Corollary 8. *The ES eigenvalues of* $K_\gamma - e$ *are*
$\frac{(\gamma-3)(\gamma-1)\sqrt{3} \pm \sqrt{3(\gamma-3)^2(\gamma-1)^2 + 8(\gamma-2)\left[(\gamma-1)^2+(\gamma-2)^2+(\gamma-1)(\gamma-2)\right]}}{2}$,
$-(\gamma-1)\sqrt{3}$ *with multiplicity* $\gamma-3$ *and* 0.

Proof. The proof is assert from above Theorem 7. The $K_\gamma - e$ is the special case of $CS_{\omega,\gamma-\omega}$. In $K_\gamma - e$, the $\omega = \gamma - 2$ and $\gamma - \omega = 2$. The eigenvalues of $K_\gamma - e$ are $-(\gamma-1)\sqrt{3}$ with multiplicity $\gamma-3$ and 0. The remaining two eigenvalues are obtain from the equitable quotient matrix of $K_\gamma - e$ is written as follows.

$$Q(K_{\gamma-e}) = \begin{pmatrix} (\gamma-3)(\gamma-1)\sqrt{3} & 2\sqrt{(\gamma-1)^2+(\gamma-2)^2 \\ & +(\gamma-1)(\gamma-2)} \\ (\gamma-2)\sqrt{(\gamma-1)^2+(\gamma-2)^2 \\ +(\gamma-1)(\gamma-2)} & 0 \end{pmatrix}.$$

The eigenvalues of $Q(K_n - e)$ are
$$\frac{(\gamma-3)(\gamma-1)\sqrt{3} \pm \sqrt{3(\gamma-3)^2(\gamma-1)^2 + 8(\gamma-2)\left[(\gamma-1)^2+(\gamma-2)^2+(\gamma-1)(\gamma-2)\right]}}{2}. \quad \square$$

Perron-Frobenius Theorem: A positive square matrix A of order γ having only one largest spectrum (Perron root) and the corresponding eigenvector has strictly positive entries.

Lemma 9. *([26]) Let A be a real $\gamma \times \gamma$ matrix with $\gamma \geq 2$, which is non-negative, symmetric, and irreducible. Let α denote the spectral radius of A, and let x be a corresponding eigenvector normalized to have unit length. This vector $\mathbf{x}$ is known as the Perron–Frobenius eigenvector. Then, A has exactly s distinct spectrum $(2 \leq s \leq \gamma)$ for real numbers $\alpha_2, \ldots, \alpha_s$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_s$, and that holds*

$$\prod_{i=2}^{s}(A - \alpha_i I) = \left(\prod_{i=2}^{s}(\alpha_1 - \alpha_i)\right)\mathbf{x}\mathbf{x}^t.$$

Since, the $M_{ES}(\Gamma)$ meets the conditions of Lemma 9. So, we have the following Lemma.

Lemma 10. *Consider a connected graph Γ that has at least two vertices. Suppose ν_1 is the spectral radius of $M_{ES}(\Gamma)$, and let $\mathbf{x}$ be the Perron–Frobenius eigenvector. Then, $M_{ES}(\Gamma)$ has exactly m distinct eigenvalues $(2 \leq m \leq n)$ for real numbers $\nu_2, \ldots, \nu_m$ such that $\nu_1 > \nu_3 > \cdots > \nu_m$, and that holds the identity*

$$\prod_{i=2}^{m}(M_{ES}(\Gamma) - \nu_i I) = \left(\prod_{i=2}^{m}(\nu_1 - \nu_i)\right)\mathbf{x}\mathbf{x}^t.$$

Lemma 11. *[6] Let Γ be a connected graph.*

1. *If Γ is the complete bipartite graph. Then the eigenvalues are symmetric about the origin.*
2. *If $\Theta(\Gamma)$ be the adjacency matrix of Γ. Then the number of different walks of length k from vertices i and j is $(i,j)^{th}$ entry of $(\Theta(\Gamma))^k$.*

By using the above Lemma, we have following result which gives the characterization of K_γ and $K_{\alpha,\beta}$ graphs from their distinct ES eigenvalues.

Theorem 12. *Consider a connected graph* Γ *on* $\gamma \geq 3$ *vertices.*

1. *The* Γ *possesses exactly two distinct* ES *eigenvalues if and only if* $\Gamma \cong K_\gamma$.

2. *The bipartite graph* Γ *has exactly three distinct* ES *eigenvalues if and only if* $\Gamma \cong K_{\alpha,\beta}$.

3. *If* Γ *has* m *distinct* ES *eigenvalues, then* $\operatorname{diam}(\Gamma) \leq m-1$.

Proof. 1. Let $\nu_1 > \nu_2$ be two distinct ES eigenvalues of Γ. Then by lemma 10,

$$M_{ES}(\Gamma) = (\nu_1 - \nu_2)\mathbf{x}\mathbf{x}^t + \nu_2 I.$$

From the above equation, the non diagonal elements of the $M_{ES}(\Gamma)$ are positive. So the graph Γ must be complete. Conversely, suppose graph Γ is complete, then $M_{ES}(\Gamma) = (\gamma-1)\sqrt{3}\Theta(\Gamma)$ thus, $\nu_i = (\gamma-1)\sqrt{3}\,\varphi_i$, i taking values from 1 to γ. Hence Γ has two distinct ES eigenvalues, $(\gamma-1)^2\sqrt{3}$ and $-(\gamma-1)\sqrt{3}$.

2. By Perron-Frobenius Theorem, $\nu_1 > 0$, ν_γ be simple and $\nu_i \neq \nu_1$, ν_γ be three distinct ES eigenvalues of Γ such that, $\nu_i \neq 0$. As Γ is connected bipartite graph, by Lemma 11 (1), Γ will have four distinct eigenvalues. It is a contradiction to $\nu_i \neq 0$ is ES eigenvalues of Γ. Therefore, $\nu_i = 0$ is $\gamma - 2$ -times ES eigenvalues of Γ. Gives Γ has three distinct ES eigenvalues viz. $\nu_1 > 0$, ν_γ and $\nu_i = 0$, $i \neq 1, \gamma$. Thus by Theorem 5 (2), graph Γ is complete bipartite. Conversely, suppose graph Γ is a complete bipartite graph with number of vertices $\gamma \geq 3$. Then it is clear that Γ is a bipartite graph and its ES matrix M_{ES} has exactly three distinct eigenvalues.

3. Let M_{ES} denote the ES matrix of the graph Γ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m$ be its m distinct eigenvalues. Suppose $\mathbf{x}$ be the unit, positive eigenvector corresponding to the largest eigenvalue ν_1. Then by Lemma 10, it follows that

$$\begin{aligned}\prod_{i=2}^{m}(M_{ES}-\nu_i I) &= (M_{ES})^{m-1}+b_1(M_{ES})^{m-2}+\cdots+b_{m-2}(M_{ES})+b_{m-1}I \\ &= \prod_{i=2}^{m}(\nu_1-\nu_i)\,\mathbf{x}\mathbf{x}^t = N.\end{aligned}$$

Observe that each entry n_{ij} of the matrix N is strictly positive for each i and j. Therefore, for any distinct vertices μ_i and μ_j there exists

a positive integer k such that $1 \leq k \leq m-1$ and $((M_{ES})^k)_{ij} > 0$. By Lemma 11 (2), this implies the existence of a walk of length k between μ_i and μ_j, and hence, a path of length at most k. Consequently, the diameter of Γ satisfies $\text{diam}(\Gamma) \leq m-1$. □

In the next result, we gives the bound for $ES(\Gamma)$ in terms of the square trace of $M_{ES}(\Gamma)$.

Theorem 13. *Let Γ be a connected graph on γ vertices. Then*

$$\frac{\text{trace}(M_{ES}^2(\Gamma))\left(\frac{\delta}{\Delta}\right)}{\sqrt{12}\Delta} \leq ES(\Gamma) \leq \frac{\text{trace}(M_{ES}^2(\Gamma))\left(\frac{\Delta}{\delta}\right)}{\sqrt{12}\delta}. \tag{1}$$

Equality holds if and only if Γ is an regular graph.

Proof. From Theorem 5 (3), we have

$$\begin{aligned}\sum_{i=1}^{\gamma} \nu_i^2 = \text{trace}(M_{ES}^2(\Gamma)) = 2(FI(\Gamma) + SZ(\Gamma)) &= 2 \sum_{\mu_i\mu_j \in E(\Gamma)} \left(\rho_i^2 + \rho_j^2 + \rho_i\rho_j\right) \\ &= 2 \sum_{\mu_i\mu_j \in E(\Gamma)} \left(\sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}\sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}\right) \\ &\leq 2\sqrt{\Delta^2 + \delta^2 + \Delta\delta} \sum_{\mu_i\mu_j \in E(\Gamma)} \sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}\end{aligned}$$

$$\begin{aligned}\text{trace}(M_{ES}^2(\Gamma)) &\leq 2\sqrt{3}\Delta\left(\frac{\Delta}{\delta}\right)ES(\Gamma) \\ \frac{\text{trace}(M_{ES}^2(\Gamma))}{2\sqrt{3}\Delta}\left(\frac{\delta}{\Delta}\right) &\leq ES(\Gamma).\end{aligned}$$

For the upper bound, we have

$$\begin{aligned}\text{trace}(M_{ES}^2(\Gamma)) &= 2 \sum_{\mu_i\mu_j \in E(\Gamma)} \left(\rho_i^2 + \rho_j^2 + \rho_i\rho_j\right) \\ = 2 \sum_{\mu_i\mu_j \in E(\Gamma)} &\left(\sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}\sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}\right)\end{aligned}$$

$$\text{trace}(M_{ES}^2(\Gamma)) \geq 2\sqrt{3}\delta\big(\frac{\delta}{\Delta}\big) \sum_{\mu_i\mu_j \in E(\Gamma)} \sqrt{\rho_i^2 + \rho_j^2 + \rho_i\rho_j}$$

$$\text{trace}(M_{ES}^2(\Gamma)) \geq 2\sqrt{3}\delta\big(\frac{\delta}{\Delta}\big) ES(\Gamma)$$

$$\frac{\text{trace}(M_{ES}^2(\Gamma))}{2\sqrt{3}\delta}\big(\frac{\Delta}{\delta}\big) \geq ES(\Gamma).$$

The equality hold if $\delta = \rho_i = \Delta$ for any $\mu_i \in V(\Gamma)$. This implies that Γ is an regular graph. □

In the next result, we gives the upper and lower bounds for spectral radius of $M_{ES}(\Gamma)$.

Theorem 14. *Let Γ be a connected graph on $\gamma \geq 1$ vertices. Then*

$$\frac{2ES(\Gamma)}{\gamma} \leq \nu_1 \leq \sqrt{\frac{2(\gamma-1)(FI(\Gamma) + SZ(\Gamma))}{\gamma}},$$

equality holds on left-hand side if and only if Γ is the regular graph and on right-hand side if and only if Γ is the complete graph.

Proof. We know that, the $M_{ES}(\Gamma)$ matrix is symmetric, irreducible and real non-negative. As Γ is a connected graph. Then by Perron-Frobenius Theorem, we have

$$\nu_1 = \max_{Y \neq 0} \frac{Y^T M_{ES} Y}{Y^T Y},$$

where Y is the any non zero real vector. Let $J = (1, 1, \ldots, 1)$ be the all-ones vector. For $Y = J$, then we obtain

$$\nu_1 \geq \frac{Y^T M_{ES} Y}{Y^T Y} = \frac{2ES(\Gamma)}{\gamma}.$$

By using Theorem 5 (1), left-hand side hold as $\sqrt{3}$ times eigenvalues of $\Theta(\Gamma)$. Therefore, $\nu_1 = \sqrt{3}\sigma$, where σ is the degree of every vertex in Γ. Also, the number of edges are $\xi = \frac{\gamma\sigma}{2}$. Then $ES(\Gamma) = \xi\sqrt{3} = \frac{\sqrt{3}\gamma\sigma}{2}$. Hence $\frac{ES(\Gamma)}{\gamma} \leq \sqrt{3}\sigma = \nu_1$. Therefore, equality hold for Γ is an regular graph.

From Theorem 5 (3), we have

$$\sum_{i=1}^{\gamma} \nu_i^2 = 2(FI(\Gamma) + SZ(\Gamma)).$$

The trace of the $M_{ES}(\Gamma)$ is 0, because the all diagonal entries of $M_{ES}(\Gamma)$ are 0. Thus, we have $\nu_1 = -\sum_{i=2}^{\gamma} \nu_i$. Now by using Cauchy-Schwarz inequality,

$$\nu_1^2 = \Big(\sum_{i=2}^{\gamma}(-\nu_i)\Big)^2 \leq (\gamma-1)\sum_{i=2}^{\gamma}\nu_i^2.$$

Hence,

$$\sum_{i=1}^{\gamma}\nu_i^2 = \nu_1^2 + \sum_{i=2}^{\gamma}\nu_i^2 \geq \nu_1^2 + \Big(\frac{\nu_1^2}{\gamma-1}\Big) = \nu_1^2\Big(1+\frac{1}{\gamma-1}\Big) = \frac{\gamma\nu_1^2}{\gamma-1}.$$

That is

$$\nu_1^2 + \sum_{i=2}^{\gamma}\nu_i^2 \geq \frac{\gamma\nu_1^2}{\gamma-1}.$$

$$\nu_1^2 \leq \frac{\gamma-1}{\gamma}\sum_{i=1}^{\gamma}\nu_i^2 = 2\Big(\frac{\gamma-1}{\gamma}\Big)(FI(\Gamma)+SZ(\Gamma))$$

then, we have

$$\nu_1 \leq \sqrt{2\Big(\frac{\gamma-1}{\gamma}\Big)(FI(\Gamma)+SZ(\Gamma))}.$$

Equality holds if $\Gamma = K_\gamma$, because degree of all vertices are equal to $\gamma-1$ and equality follows directly from Theorem 5 (1). □

Theorem 15. *Let Γ be a connected graph on $\gamma \geq 1$ vertices, ν_1 and $\overline{\nu}_1$ be the largest eigenvalue of Γ and $\overline{\Gamma}$. Then*

$$\nu_1 + \overline{\nu}_1 \leq \Delta\sqrt{\frac{6(\gamma-1)\xi}{\gamma}} + (\gamma-1-\delta)\sqrt{\frac{3(\gamma-1)\big(\gamma(\gamma-1)-2\xi\big)}{\gamma}},$$

equality holds if and only if Γ is the complete graph.

Proof. From the Theorem 14, we have

$$\nu_1 + \overline{\nu}_1 \leq \sqrt{\frac{2(\gamma-1)(FI(\Gamma)+SZ(\Gamma))}{\gamma}} + \sqrt{\frac{2(\gamma-1)(FI(\overline{\Gamma})+SZ(\overline{\Gamma}))}{\gamma}}$$

$$\leq \sqrt{\frac{2(\gamma-1)}{\gamma}\sum_{\mu_i\mu_j\in E(\Gamma)}\big(\rho_i^2+\rho_j^2+\rho_i^2\rho_j^2\big)} + \sqrt{\frac{2(\gamma-1)}{\gamma}\sum_{\mu_i\mu_j\in E(\overline{\Gamma})}\big(\bar{\rho}_i^2+\bar{\rho}_j^2+\bar{\rho}_i^2\bar{\rho}_j^2\big)},$$

where μ_i be the vertex in Γ and $\overline{\Gamma}$ and ρ_i and $\overline{\rho}_i$ be its degree respectively. As $\delta \leq \rho_i \leq \Delta$, we have

$$\begin{aligned}
&\leq \sqrt{\frac{2(\gamma-1)}{\gamma}\sum_{\mu_i\mu_j\in E(\Gamma)}\big(3\Delta^2\big)} + \sqrt{\frac{2(\gamma-1)}{\gamma}\sum_{\mu_i\mu_j\in E(\overline{\Gamma})}3\big(\gamma-1-\delta\big)^2} \\
&= \sqrt{\frac{2(\gamma-1)}{\gamma}\big(3\xi\Delta^2\big)} + \sqrt{\frac{2(\gamma-1)}{\gamma}\left(\frac{\gamma(\gamma-1)}{2}-\xi\right)3\big(\gamma-1-\delta\big)^2} \\
&= \Delta\sqrt{\frac{6(\gamma-1)\xi}{\gamma}} + \big(\gamma-1-\delta\big)\sqrt{\frac{3(\gamma-1)}{\gamma}\Big(\gamma(\gamma-1)-2\xi\Big)}
\end{aligned}$$

Conversely, from Theorem 5 (1), easily we get desire result. □

Theorem 16. *Let Γ be a graph on $\gamma \geq 1$ vertices and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_\gamma$ be the ES eigenvalues. Then $|\nu_1| = |\nu_2| = \cdots = |\nu_\gamma|$ if and only if $G \cong \overline{K_\gamma}$ or $G \cong \frac{\gamma}{2}K_2$, where γ is even.*

Proof. Suppose that $|\nu_1| = |\nu_2| = \cdots = |\nu_\gamma|$. If the number of isolated vertices in Γ are greater than equal to one, then all eigenvalues of Γ are zero. i.e. $\nu_1 = \nu_2 = \cdots = \nu_\gamma = 0$, so this implies that $\Gamma \cong \overline{K_\gamma}$. If Γ does not consist of any isolated vertex and every vertex in Γ is a pendent vertex, then Γ is perfect matching. i.e. $\Gamma \cong \frac{\gamma}{2}K_2$. In this case, all the eigenvalues have the same modulus. Otherwise, Γ contains a connected component Γ' at least three vertices, then by the Perron-Frobenius Theorem, again we get contradiction, because the largest eigenvalue in modulus is strictly greater than the others. Conversely, we easily check that for $\overline{K_\gamma}$ and $\frac{\gamma}{2}K_2$ all its ES eigenvalues are same modulus. □

3 ES-Energy ($\mathcal{E}_{ES}$)

The graph energy plays an important role in spectral graph theory and chemical graph theory. Graph energy is widely used to study molecular stability, chemical bonding, network structure, and graph properties. Due to its applications in chemistry, physics, computer science, and network analysis, graph energy has attracted considerable attention from researchers. For more details, readers may refer to [15, 16, 22, 5, 31].

In this section, we obtain the ES energy of regular graphs and complete bipartite graph. We also establish upper and lower bounds for the ES energy of graphs in terms of the Forgotten index and the second Zagreb index.

Lemma 17 ([12]). *Let the matrices A_1 and A_2 be of order $\gamma \times \gamma$. Then*

$$\sum_{i=1}^{\gamma} \lambda_i(A_1 + A_2) \leq \sum_{i=1}^{\gamma} \lambda_i(A_1) + \sum_{i=1}^{\gamma} \lambda_i(A_2),$$

where λ_i denotes the eigenvalue of the matrix A. Moreover, the equality hold if and only if there exists an orthogonal matrix A, so that AA_1 and AA_2 are positive semi-definite matrices.

Theorem 18. *Let Γ be a connected graph on $\gamma \geq 3$ vertices.*

1. *If Γ is an σ-regular graph. Then $\mathcal{E}_{ES}(\Gamma) = \sigma\sqrt{3}\ \mathcal{E}_{\Theta}(\Gamma)$. In particular, if $\Gamma = K_\gamma$, then $\mathcal{E}_{ES}(K_\gamma) = \sqrt{3}(\gamma - 1)\mathcal{E}_{\Theta}(K_\gamma)$. Similarly, if $\Gamma = C_\gamma$, then $\mathcal{E}_{ES}(C_\gamma) = 2\sqrt{3}\mathcal{E}_{\Theta}(C_\gamma)$.*
2. *If Γ is a complete bipartite graph of order $(\alpha + \beta = \gamma)$, then $\mathcal{E}_{ES}(\Gamma) = \sqrt{\alpha^2 + \beta^2 + \alpha\beta}\ \mathcal{E}_{\Theta}(K_{\alpha,\beta})$.*
3. *Let the $FI(\Gamma)$ and $SZ(\Gamma)$ be the Forgotten index and the second Zagreb index of graph Γ. Then*
   (i) $\mathcal{E}_{ES}(\Gamma) \geq 2\sqrt{FI(\Gamma) + SZ(\Gamma)}$ and
   (ii) $\mathcal{E}_{ES}(\Gamma) \leq \sqrt{2\gamma(FI(\Gamma) + SZ(\Gamma))}$.
4. *If Γ is a path graph. Then $\mathcal{E}_{ES}(P_\gamma) \leq \sqrt{12}\ \mathcal{E}_{\Theta}(P_\gamma) + 2(\sqrt{7} - \sqrt{12})$.*

Proof. 1. The proof is directly follows from the Theorem 5 (1).

2. From the Theorem 5 (2), easily we get the result.

3. By the definition of ES energy, $\mathcal{E}_{ES}$ and Theorem 5 (3), we have

$$\begin{aligned}
(\mathcal{E}_{ES}(\Gamma))^2 &= \sum_{i=1}^{\gamma} |\nu_i|^2 + 2 \sum_{1 \leq i < j \leq \gamma} |\nu_i \nu_j| \\
&\geq \sum_{i=1}^{\gamma} \nu_i^2 + 2 \left| \sum_{1 \leq i < j \leq \gamma} \nu_i \nu_j \right| \\
&= 2(FI(\Gamma) + SZ(\Gamma)) + 2|-(FI(\Gamma) + SZ(\Gamma))| \\
&= 4(FI(\Gamma) + SZ(\Gamma)).
\end{aligned}$$

This implies that, $\mathcal{E}_{ES}(\Gamma) \geq 2\sqrt{(FI(\Gamma) + SZ(\Gamma))}$.

4. By the definition of ES energy $\mathcal{E}_{ES}$ and Theorem 5 (3), we have

$$(\mathcal{E}_{ES}(\Gamma))^2 = \sum_{i=1}^{\gamma} |\nu_i|^2 + 2 \sum_{1 \leq i < j \leq \gamma} |\nu_i \nu_j|.$$

Using suitable bounds on the above terms, we get:

$$(\mathcal{E}_{ES}(\Gamma))^2 \leq 4(FI(\Gamma) + SZ(\Gamma)) \leq 2\gamma(FI(\Gamma) + SZ(\Gamma)).$$

Taking square roots on both sides yields the result:

$$\mathcal{E}_{ES}(\Gamma) \leq \sqrt{2\gamma(FI(\Gamma) + SZ(\Gamma))}.$$

5. The Euler Sombor matrix of P_n can be written as,

$$M_{ES}(P_\gamma) = (\sqrt{12}) \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

$$+(\sqrt{7} - \sqrt{12}) \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

$$i.e.\ M_{ES}(P_\gamma) = (\sqrt{12})\ \Theta(P_\gamma) + B$$

Now the ES energy is, $\mathcal{E}_{ES}(P_\gamma) = \sqrt{12}\mathcal{E}_{ES}(P_\gamma) + \mathcal{E}(B)$. But the Euler Sombor eigen values of B are $\pm\{\sqrt{7} - \sqrt{12}\}$. Then using Lemma 17, we have $\mathcal{E}_{ES}(P_\gamma) \leq \sqrt{12}\ \mathcal{E}_{\Theta}(P_\gamma) + 2(\sqrt{7} - \sqrt{12})$.

□

Corollary 19. *Let Γ be a connected graph on $\gamma \geq 3$ and m edges. Then $2\sqrt{FI(\Gamma) + SZ(\Gamma)} \leq \mathcal{E}_{ES}(\Gamma) \leq \sqrt{2\gamma(FI(\Gamma) + SZ(\Gamma))}$, where $FI(\Gamma)$-Forgotten index and $SZ(\Gamma)$-second Zagreb index of graph Γ.*

4 Correlation

In this section, we compute ES, FI and SZ energies of octane isomers by using the Python programming. Also, we find the correlation of ES energy with FI and SZ energies of octane isomers. In addition, we obtain the correlation of ES, FI and SZ energies with respect to the physicochemical properties of octane isomers viz. boiling point(BP), entropy, acentric factor (AF), enthalpy of vaporization (HVAP) and standard enthalpy of vaporization (DHVAP).

Table 1: Euler Sombor, Forgotten and second Zagreb energies of octane isomers

Sr. No.	Molecule	$\mathcal{E}_{ES}$	$\mathcal{E}_{FI}$	$\mathcal{E}_{SZ}$
1	octane	30.2172	66.2439	31.6212
2	2-methyl-heptane	30.298	75.489	32.0568
3	3-methyl-heptane	31.61	77.6835	33.8232
4	4-methyl-heptane	30.194	75.0974	32.9665
5	3-ethyl-hexane	31.415	76.9651	34.5159
6	2,2-dimethyl-hexane	32.394	99.8456	33.9879
7	2,3-dimethyl-hexane	31.654	86.9688	35.5943
8	2,4-dimethyl-hexane	31.452	86.2309	34.114
9	2,5-dimethyl-hexane	31.642	87.171	33.3169
10	3,3-dimethyl-hexane	32.464	100.2934	36.4399
11	3,4-dimethyl-hexane	32.98	89.1456	37.1639
12	2-methyl-3-ethyl-pentane	31.502	86.293	36.0791
13	3-methyl-3-ethyl-pentane	33.928	102.0731	38.1993
14	2,2,3-trimethyl-pentane	33.804	111.5061	38.7971
15	2,2,4-trimethyl-pentane	31.964	107.718	33.9633
16	2,3,3-trimethyl-pentane	33.976	111.8089	39.7411
17	2,3,4-trimethyl-pentane	32.98	98.451	37.7739
18	2,2,3,3-tetramethyl-butane	34.632	134.0448	42.332

Table 2: Correlation Coefficient of ES energy to FI and SZ energies of octane isomers

Sr.No.	Energy	Correlation Coefficient
1	$\mathcal{E}_{FI}$	0.8900
2	$\mathcal{E}_{SZ}$	0.9310

Table 3: Some Physicochemical properties of octane isomers

Sr. No.	Molecule	BP	Entropy	AF	HVAP	DHVAP
1	octane	125.7	111.7	0.3979	73.19	9.915
2	2-methyl-heptane	117.6	109.8	0.3792	70.3	9.484
3	3-methyl-heptane	118.9	111.3	0.371	71.3	9.521
4	4-methyl-heptane	117.7	109.3	0.3715	70.91	9.483
5	3-ethyl-hexane	118.5	109.4	0.3625	71.7	9.476
6	2,2-dimethyl-hexane	106.8	103.4	0.3394	67.7	8.915
7	2,3-dimethyl-hexane	115.6	108	0.3483	70.2	9.272
8	2,4-dimethyl-hexane	109.4	107	0.3442	68.5	9.029
9	2,5-dimethyl-hexane	109.1	105.7	0.3568	68.6	9.051
10	3,3-dimethyl-hexane	112	104.7	0.3226	68.5	8.973
11	3,4-dimethyl-hexane	117.7	106.6	0.3404	70.2	9.316
12	2-methyl-3-ethyl-pentane	115.6	106.1	0.3324	69.7	9.209
13	3-methyl-3-ethyl-pentane	118.3	101.5	0.3069	69.3	9.081
14	2,2,3-trimethyl-pentane	109.8	101.3	0.3001	67.3	8.826
15	2,2,4-trimethyl-pentane	99.24	104.1	0.3054	64.87	8.402
16	2,3,3-trimethyl-pentane	114.8	102.1	0.2932	68.1	8.897
17	2,3,4-trimethyl-pentane	113.5	102.4	0.3174	68.37	9.014
18	2,2,3,3-tetramethyl-butane	106.5	93.06	0.2553	66.2	8.41

Table 4: Correlation coefficients of BP, Entropy, AF, HVAP and DHVAP of octane isomers with energy

Sr. No.	Energy	BP	Entropy	AF	HVAP	DHVAP
1	$\mathcal{E}_{ES}$	-0.374	-0.8740	-0.9003	-0.6129	-0.6966
2	$\mathcal{E}_{FI}$	-0.672	-0.9611	-0.9743	-0.532	-0.9105
3	$\mathcal{E}_{SZ}$	-0.248	-0.8557	-0.8967	-0.5053	-0.6139

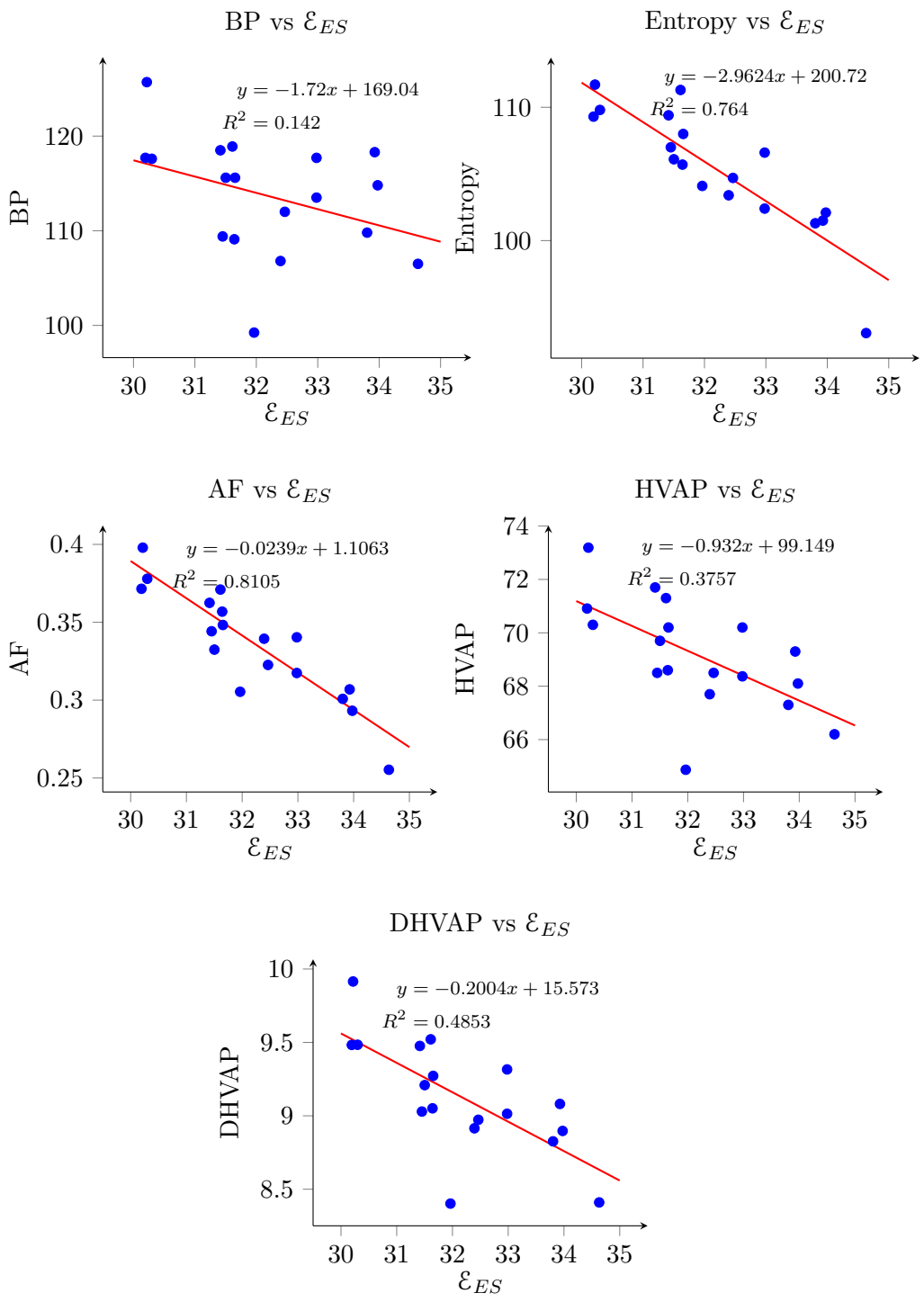

Figure 1: Linear Regression plots for BP, Entropy, AF, HVAP and DHVAP with respect to $\mathcal{E}_{ES}$

5 Conclusion

In this work, we determined the eigenvalues of certain families of graphs, and using these eigenvalues we characterized graphs such as complete and bipartite graphs. We also obtained bounds for the ES spectral radius and the ES index in terms of the Forgotten and second Zagreb indices. In Section 3, several results concerning the Euler–Somber energy of graphs are presented.

In the final section, we computed the ES, FI, and SZ energies of octane isomers. We attempted to establish correlations between the FI and SZ energies with respect to the ES energy of these octane isomers. Additionally, we examined the correlations between the ES, FI, and SZ energies and various physicochemical properties of octane isomers. Data from Table 2 indicate that the SZ energy and FI energy exhibit the strongest relationships with the ES energy, although the correlation for FI energy is slightly weaker. Furthermore, Table 4 reveals a very strong relationship between FI energy, entropy, and the acentric factor. Overall, the FI energy emerges as the most reliable predictor of the physical behavior of octane molecules, as it consistently demonstrates the highest correlation across most categories.

In future work, it would be possible to explore additional properties of this index. It may also be possible to identify useful connections between the results obtained from this index and the chemical properties of molecules.

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